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Compressibility Effects in Aerodynamics

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Introduction

A FEW DECADES ago the most enthusiastic admirer of mathematical analysis would not have expected that practical engineers engaged in the design of aircraft would have so much use for the mathematical theory of fluid motion as is the case in modern aeronautical engineering. The various applications of the theory are based almost entirely on the hydrodynamics of incompressible perfect fluids, a discipline that half a century ago was considered as an interesting field of pure science having very little to do with the motion which actually takes place in a real fluid. Recently, interest has been centered on another branch of fluid mechanics, namely, on the mechanics of compressible fluids. The aeronautical engineer is pounding hard on the closed door leading into the field of supersonic motion. He realizes that the price that has to be paid for further increase of speed becomes higher and higher if he nears this frontier. However, he wonders whether the mathematical theory could not be used as a guide for avoiding a premature drop of aerodynamic efficiency. The present paper has the objective of reviewing the present status of the theory of compressible fluids from the practical standpoint of its usefulness for interpretation of experimental research and guidance in design.

Irrotational Motion of Ideal Fluids

The problem of irrotational motion of ideal incompressible fluids is more or less a problem of pure geometry. Every velocity field that satisfies the conditions of no vorticity (zero rotation) and the condition of continuity (zero divergence) represents a possible, *i.e.*, dynamically correct, flow pattern of an ideal incompressible fluid. For example, in the case of two-dimensional motion, any network of two families of orthogonal curves represents a network of streamlines and equipotential lines of a dynamically correct flow pattern provided it satisfies one condition: that all its elements are of square shape. The flow is completely determined by the rule that the amount of fluid passing between two neighboring streamlines is constant and the "potential difference" between two successive equipotential lines is a constant quantity. If

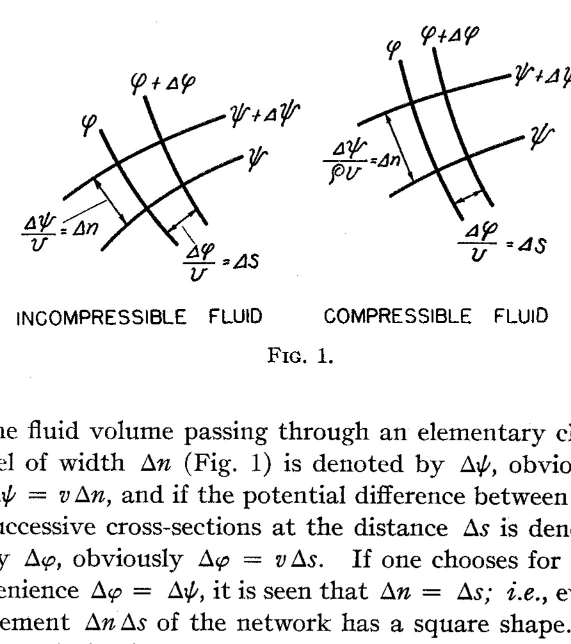

FIG. 1.

the fluid volume passing through an elementary channel of width Δn (Fig. 1) is denoted by $\Delta\psi$, obviously $\Delta\psi = v\,\Delta n$, and if the potential difference between two successive cross-sections at the distance Δs is denoted by $\Delta\varphi$, obviously $\Delta\varphi = v\,\Delta s$. If one chooses for convenience $\Delta\varphi = \Delta\psi$, it is seen that $\Delta n = \Delta s$; *i.e.*, every element $\Delta n\,\Delta s$ of the network has a square shape. If the velocity increases or decreases along a streamline, the elements of the network shrink or expand in such a way that both Δs and Δn are inversely proportional to the velocity. Hence the elements of the network always keep their square shapes.

The pressure does not enter the problem directly, because in the case of incompressibility the pressure is not related to the physical state of the fluid. This can be illustrated, for example, by the fact that provided the fluid has no free surface (*i.e.*, a surface of given constant pressure) one can superpose or remove a gravity

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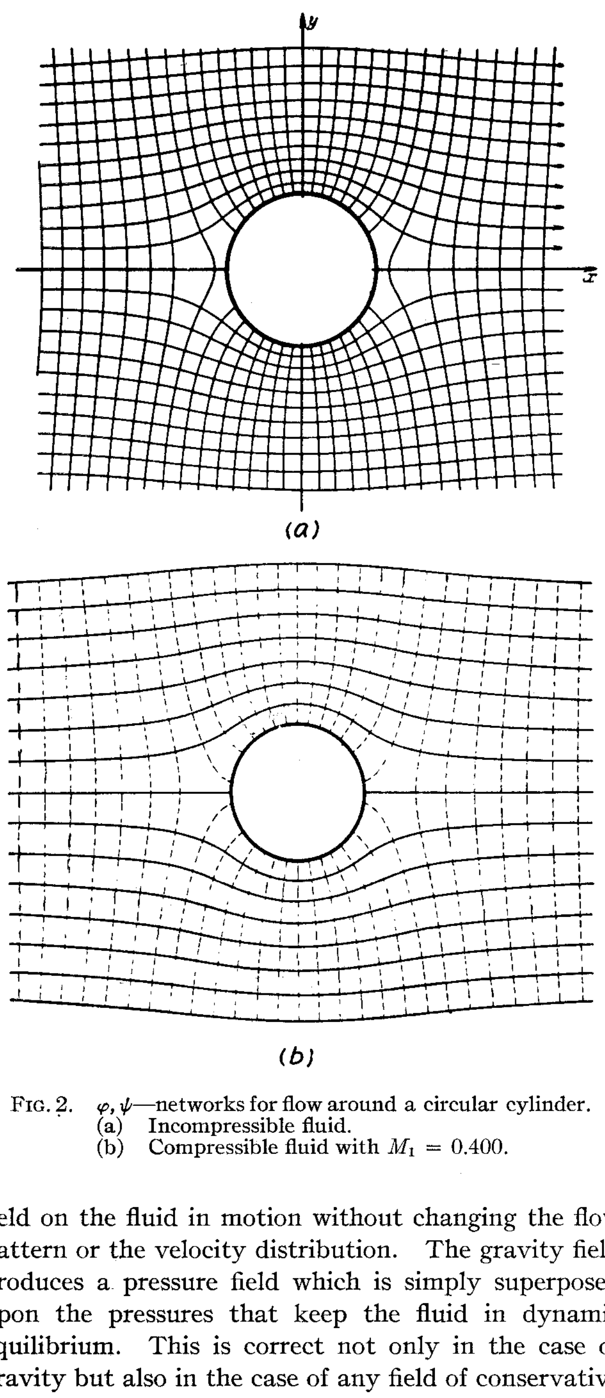

FIG. 2. φ, ψ—networks for flow around a circular cylinder.
(a) Incompressible fluid.
(b) Compressible fluid with $M_1 = 0.400$.

field on the fluid in motion without changing the flow pattern or the velocity distribution. The gravity field produces a pressure field which is simply superposed upon the pressures that keep the fluid in dynamic equilibrium. This is correct not only in the case of gravity but also in the case of any field of conservative forces.

In dealing with compressible fluids two new aspects have to be taken into account:

(a) The continuity condition for the channels between neighboring streamlines involves the variable density; namely, one has to write:

$$\Delta\psi = \rho v \Delta n \tag{1}$$

where $\Delta\psi$ is now the fluid mass passing through the channel (Fig. 1). The potential difference is equal to $\Delta\varphi = v\Delta s$ as in the case of incompressible fluid, and therefore in a network drawn with a constant ratio between $\Delta\varphi$ and $\Delta\psi$ the ratio $\Delta s/\Delta n$ is variable and proportional to the density ρ. Consequently the elements of the network are elongated in the flow direction if the density increases, and expanded normal to the flow direction if the density decreases. This is clearly seen in Fig. 2 where the φ, ψ networks for the flow of an incompressible and a compressible fluid around a circular cylinder are plotted.

(b) The flow problem cannot be solved by means of geometrical considerations alone. The density is connected with the pressure and temperature by the equation of state. If heat conduction is neglected and it is assumed that all particles have started from the same state at rest, the density and pressure are connected by the "isentropic relation"

$$p = \text{const. } \rho^{\gamma} \tag{2}$$

where γ is the ratio of the two specific heats (for constant pressure and constant volume) of the fluid. This relation, combined with Bernoulli's equation

$$vdv + (1/\rho)dp = 0 \tag{3}$$

furnishes a relation between density and velocity. This relation has to be introduced into the geometrical conditions of irrotationality and continuity in order to solve the flow problem.

THERMODYNAMIC RELATIONS

If p_0, ρ_0 denote the pressure and density of the fluid at rest, Eq. (2) becomes

$$p = (p_0/\rho_0^{\gamma})\rho^{\gamma} \tag{4}$$

On the other hand, Eq. (3) can be written in the form

$$vdv + (1/\rho)(dp/d\rho)d\rho = 0 \tag{5}$$

The quantity $dp/d\rho$ is equal to the square of the velocity of sound; *i.e.*, the velocity of propagation of infinitesimally small pressure changes. With the notations $dp/d\rho = a^2$ and $(dp/d\rho)_0 = \gamma(p_0/\rho_0) = a_0^2$ (the velocity of sound at rest) the following relations are easily obtained:

$$a^2 = a_0^2 - [(\gamma - 1)/2]v^2 \tag{6}$$

$$\rho = \rho_0\left[1 - \frac{\gamma-1}{2}\frac{v^2}{a_0^2}\right]^{\frac{1}{\gamma-1}} = \rho_0\left[1 + \frac{\gamma-1}{2}\frac{v^2}{a^2}\right]^{-\frac{1}{\gamma-1}} \tag{7}$$

$$p = p_0\left[1 - \frac{\gamma-1}{2}\frac{v^2}{a_0^2}\right]^{\frac{\gamma}{\gamma-1}} = p_0\left[1 + \frac{\gamma-1}{2}\frac{v^2}{a^2}\right]^{-\frac{\gamma}{\gamma-1}} \tag{8}$$

$$\frac{v^2}{2} = \frac{a_0^2}{\gamma-1}\left[1 - \left(\frac{\rho}{\rho_0}\right)^{\gamma-1}\right] = \frac{a^2}{\gamma-1}\left[\left(\frac{\rho_0}{\rho}\right)^{\gamma-1} - 1\right] \tag{9}$$

$$\frac{v^2}{2} = \frac{a_0^2}{\gamma-1}\left[1 - \left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{\gamma}}\right] = \frac{a^2}{\gamma-1}\left[\left(\frac{p_0}{p}\right)^{\frac{\gamma-1}{\gamma}} - 1\right] \tag{10}$$

These relations are well known and are stated here only for the reader's convenience. From (6) and (10) it follows that the so-called critical velocity, *i.e.*, the value of v which is equal to the velocity of sound corresponding to the same pressure and density, and the pressure corresponding to this velocity are given by

$$v_{cr} = \sqrt{2/(\gamma + 1)}\, a_0 \quad p_{cr} = 2/(\gamma + 1)^{\gamma/\gamma - 1}\, p_0 \quad (11)$$

For air $\gamma = 1.405$; therefore $v_{cr} = 0.913\, a_0$ and $p_{cr} = 0.527\, p_0$.

For moderate values of v/a_0 the relations (8) can be approximated by the formulas

$$p = p_0 - (\rho_0 v^2/2)\,[1 - (v^2/4a_0^2)]$$

and

$$p = p_0 - (\rho v^2/2)[1 + (v^2/4a^2)]$$

The last equation can be used, for example, to estimate the error made by using for the dynamic pressure $p_0 - p$ the value $\rho v^2/2$ as in the case of incompressible fluids.

Mechanical and Electrical Analogies

(a) Consider the flow of a shallow layer of a heavy, ideal and incompressible fluid. The fluid is bounded by a horizontal plane bottom surface, by vertical walls, and by an upper free surface of constant pressure. The flow of water in a broad channel or in a river is a good example of such a flow, provided the frictional resistance is neglected or it is assumed that the bottom is slightly inclined and the component of gravity in the direction of the flow compensates for the frictional resistance. (This latter assumption can only be realized as far as the average resistance is concerned.) If the variable depth of the fluid layer is denoted by h and the velocity averaged over a vertical line by v, the continuity equation for stationary flow can be written in the form

$$hv\,\Delta n = \Delta\psi = \text{const.} \quad (12)$$

where Δn is the distance between neighboring streamlines. On the other hand, Bernoulli's equation requires that

$$(v^2/2g) + h = \text{const.}$$

Therefore

$$vdv + gdh = 0 \quad (13)$$

Denote the quantity $gh^2/2$ by P; then Eq. (13) can be written in the form

$$vdv + (1/h)dP = vdv + (1/h)(dP/dh)dh = 0 \quad (14)$$

It is seen that the two-dimensional flow of a layer of a compressible fluid of constant depth can be compared with the flow of a layer of variable depth of an incompressible fluid. The Eqs. (12) and (14) are identical with the Eqs. (1) and (3) of a compressible fluid if the depth h is replaced by the density ρ and the quantity P by the pressure p. As a matter of fact, P is equal to the resultant of the pressure forces acting over a section of unit width and depth h for an incompressible fluid of unit density. Furthermore, $dP/dh = gh$ is equal to the square of the wave velocity of long waves propagating in a shallow channel of depth h. This wave velocity takes the place of the velocity of sound in the gas. The so-called hydraulic jump corresponds to the shock wave.

However, the relation between pressure and density of air is different from the relation between P and h. For isentropic change of state, $p = \text{const.}\ \rho^\gamma$, whereas $P = \text{const.}\ h^2$. It is seen that the analogy would be perfect for $\gamma = 2$; however, for air $\gamma = 1.40$ and the highest value of γ (for a monatomic gas) is 1.66. Nevertheless, the analogy is interesting for qualitative information on possible flow patterns, especially in the case of supersonic flow of a gas, which corresponds to the so-called supercritical flow in a channel or river ("torrent"). Many interesting applications have been made by several authors (cf. bibliography at the end of this paper). It seems that E. Jouguet was the first to point out the analogy.

(b) It is known that two-dimensional flow problems for incompressible fluids can be solved by use of a so-called "electrolytic bath" consisting of a layer of liquid of appropriate electrolytic conductivity. Assume that an alternating current passes through the liquid, and denote the amplitude of the alternating voltage by V, the intensity of the alternating current by I, and the conductivity of the liquid by σ. Then the relation between I and the derivative of the voltage in the direction of the current vector is given by

$$I = \sigma(\partial V/\partial s) \quad (15)$$

On the other hand, from the continuity of the electric flux it follows that

$$Ih\,\Delta n = \text{const.} \quad (16)$$

where h is the depth of the liquid layer and Δn is the normal distance between neighboring lines drawn orthogonal to the lines of constant voltage.

It is seen that the Eqs. (15) and (16) are analogous to the fundamental equations for irrotational flow of an incompressible fluid if V/σ is replaced by the velocity potential and I by the velocity v. If h is variable, Eq. (16) is identical with the continuity equation for a gas whose density at an arbitrary point is proportional to the value of h at the same point. Hence the "electrolytic bath" can be used for solution of problems of compressible flow by means of an iteration method. One can first solve the problem for $h = \text{const.}$ by measuring, for example, the V distribution for certain given boundary conditions; then ρ can be calculated for the entire field as function of v, and h modified in such a way that $h = \text{const.}\ \rho$. By carrying out the experiment for the modified shape of the setup one obtains a corrected distribution of V and the procedure can be repeated again.

G. I. Taylor and C. F. Sharman have carried out such experiments; the bottom for the liquid layer was made from an insulating material that could easily be carved out to an arbitrary shape. Of course, if the gradient of h is large, the flow of electricity is no longer two-dimensional, and therefore the approximation furnished by the electric experiment is not very good. This occurs especially if the velocity approaches the velocity of sound.

One-Dimensional Gasdynamics

Certain flow problems of compressible fluids can be treated as one-dimensional problems. For example, the flow of a gas through a duct with slightly variable cross-section area can be computed with good approximation by introducing mean values for velocity, pressure and density, the average values being taken over cross-sections of the duct. This method of treatment is similar to the method generally used in hydraulics. Therefore, the theory presented in this section can be called "gasdynamics in hydraulic fashion." Problems in aerodynamics usually do not permit the use of such simplified methods; nevertheless, the one-dimensional case is of interest to the aerodynamicist, since it gives an opportunity to recognize the essential differences between the mechanics of incompressible and compressible fluids.

If v denotes the mean velocity and ρ the mean density of the fluid in an arbitrary cross-section of area S, the flux of mass passing through this cross-section is equal to $S\rho v$. The conservation of matter or the continuity of flow requires that in the stationary case this quantity be constant for all cross-sections and that in the non-stationary case

$$S(\partial\rho/\partial t) + (\partial/\partial x)(S\rho v) = 0 \qquad (17)$$

The coordinate x is taken along the axis of the duct. In the case of stationary flow the velocity and density are connected by the relation (5) (Bernoulli's equation). From the continuity equation $S\rho v = \text{const.}$ it follows that

$$(dS/S) + (d\rho/\rho) + (dv/v) = 0 \qquad (18)$$

Substituting from (5) the value of $(d\rho/\rho) = -(vdv/a^2)$, one obtains

$$(d\rho/\rho) + (dv/v) = [1 - (v^2/a^2)](dv/v) \qquad (19)$$

The ratio v/a is called Mach's number M. Then Eq. (18) becomes

$$dS/S = -[(d\rho/\rho) + (dv/v)] = -(1 - M^2)(dv/v) \qquad (20)$$

It is seen that if $M < 1$ the cross-section area of the duct decreases when the velocity increases as in the case of an incompressible fluid; whereas for $M > 1$ the cross-section area increases with increasing velocity. The condition $M = 1$ involves $dS = 0$. For example, in a converging-diverging nozzle the velocity of sound can occur only in the throat section.

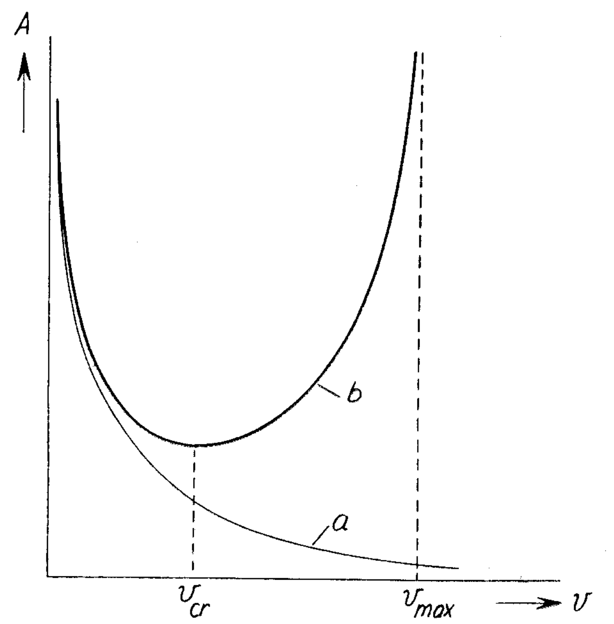

Fig. 3. Relation between the cross-sectional area A and the flow velocity in a nozzle.
(a) Incompressible fluid.
(b) Compressible fluid.

Fig. 3 shows the distribution of cross-section area as function of the velocity v, first for an incompressible fluid (line a), then for a compressible fluid (line b) according to Eq. (20). Assume that a certain nozzle (Fig. 4) is designed for a pressure ratio (p_0/p_e), where p_e is smaller than the critical value p_{cr} given by Eq. (11). In this case the local velocity of sound is reached in the throat cross-section. Then there are two possibilities of flow with isentropic change of state. The gas might expand to the pressure p_e, flowing with supersonic velocity, or it might be compressed isentropically. In the second case the flow is subsonic and the pressure reaches the value p_e' at the exit. The velocity and pressure at the exit are the same as in the cross-section of the same area in the converging portion of the nozzle. The mass flow of gas through the nozzle in the two cases is the same. If the exit pressure is greater than p_e', *i.e.*, it is between p_0 and p_e', the amount of flow varies with the pressure difference between entrance and exit. If the pressure at the exit varies between p_e and p_e', the capacity is constant.

However, the question arises: What happens with the gas if the pressure is varied between these limits? Observation shows that in this case a more or less sudden change of pressure and velocity occurs somewhere. Assume that this shock occurs in a certain cross-section of the nozzle and that the change of state downstream of the shock is isentropic again; then the gas must have a higher value of the entropy than in front of the shock. The possible pressure distribution for constant values of the entropy which are consistent with the given cross-section areas and the given capacity can easily be plotted (Fig. 4). The upper branches of such pressure curves correspond to flows with subsonic, the lower

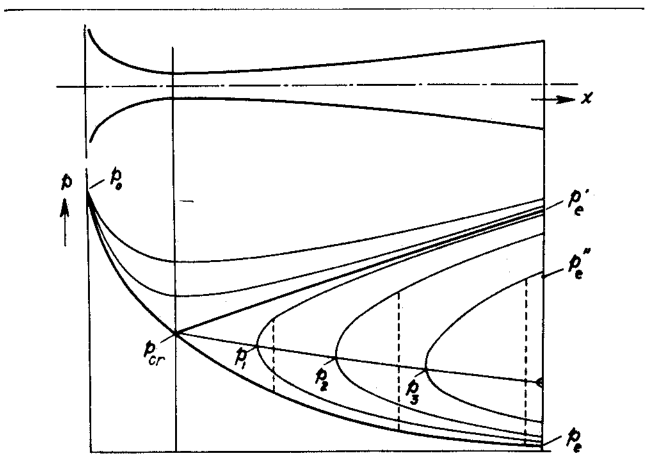

FIG. 4. Pressure distribution along a Laval nozzle.

branches to flows with supersonic velocity. It can be shown that in the case of such a shock the conditions for the conservation of matter, momentum and energy can be satisfied only if the transition takes place between supersonic and subsonic flow. Hence, a jump is physically possible only between points on the initial curve and points on the upper branch of one of the other curves. The location of the shock is determined by the three conditions mentioned. If the end pressure is slightly lower than p_e', the shock occurs close to the throat; if the pressure is lowered, the shock moves toward the exit. There is a value of the end pressure p_e'' that produces a shock right at the exit. If the exit pressure is smaller than p_e'' but greater than p_e, the shock occurs in the free jet in the form of an "oblique" shock wave.

It must be noted that so far the fluid has been considered compressible but non-viscous. Therefore, the results certainly will be modified by viscosity and especially by boundary layer effects. When the nozzle acts as diffusor, *i.e.*, when the flow occurs in the direction of adverse pressure gradient, separation may occur and change the character of the flow and the pressure distribution. However, even if such effects are neglected for the time being, the following conclusions are of interest in view of the general multi-dimensional flow problems discussed in the following sections. In the case of an incompressible fluid, if the pressure at the entrance and exit of a nozzle is arbitrarily given, there always exists a flow with a continuous pressure and velocity distribution which satisfies the conditions of continuity and dynamic equilibrium, the amount of flow being a function of the given pressure difference. In the case of a compressible fluid this is only true if the pressure difference is under a certain limit. Beyond a certain limiting value of the pressure difference the velocity in the nozzle exceeds the velocity of sound and a continuous solution exists only for one definite value of the pressure difference. It appears that in the case of flows that are partially subsonic, partially supersonic, the existence of continuous velocity fields is one of the questions of basic importance.

STATIONARY FLOW AROUND A BODY

The main problem of primary interest for aeronautical applications is that of the stationary flow around a body. If the body is approximately of cylindrical shape, like an airplane wing, the flow can be considered as two-dimensional. The flow around a nacelle, fuselage or cowling offers examples of typical three-dimensional problems. In some cases the assumption of axial symmetry simplifies the problem.

A multi-dimensional flow can be considered as consisting of elementary channels of infinitesimal cross-section area. For each of these channels the continuity equation and Bernoulli's equation must hold. However, since the channels are not separated by solid walls, the conditions of dynamic equilibrium must be satisfied in directions normal to the flow as well as along the channel. If the flow at infinite distance from the body is parallel and uniform and the effect of viscosity can be neglected, these equilibrium conditions are identically satisfied by the existence of a velocity potential, *i.e.*, by the fact that the flow is irrotational in the whole field.

Using Cartesian coordinates x_1, x_2, x_3, and denoting the velocity components by v_1, v_2, v_3, the continuity equation which corresponds to Eq. (17) can be written in the form

$$\frac{\partial \rho}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(\rho v_i) = 0 \qquad (21)$$

For stationary flow $\partial\rho/\partial t = 0$ and therefore

$$\sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i} + \frac{1}{\rho}\sum_{i=1}^{3} v_i \frac{\partial \rho}{\partial x_i} = 0 \qquad (22)$$

For irrotational flow it follows from Bernoulli's equation that for $i = 1, 2$ and 3

$$\frac{\partial p}{\partial x_i} = \frac{dp}{d\rho}\frac{\partial \rho}{\partial x_i} = a^2 \frac{\partial \rho}{\partial x_i} = -\frac{\rho}{2}\frac{\partial v^2}{\partial x_i} \qquad (23)$$

where $v^2 = v_1^2 + v_2^2 + v_3^2$. Substituting the value of $\partial\rho/\partial x_i$ from Eq. (23) in Eq. (22) the following relation is obtained:

$$\sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i} = \frac{1}{a^2}\sum_{i=1}^{3}\sum_{k=1}^{3} v_i v_k \frac{\partial v_i}{\partial x_k} \qquad (24)$$

Introducing the velocity potential φ, defined by the relations $v_i = \partial\varphi/\partial x_i$, one obtains

$$\Delta\varphi = \sum_{i=1}^{3} \frac{\partial^2 \varphi}{\partial x_i^2} = \frac{1}{a^2}\sum_{i=1}^{3}\sum_{k=1}^{3} \frac{\partial \varphi}{\partial x_i}\frac{\partial \varphi}{\partial x_k}\frac{\partial^2 \varphi}{\partial x_i \partial x_k} \qquad (25)$$

It is seen that in the case of an incompressible fluid ($a = \infty$) the right side of the equation vanishes and φ is determined by the equation of Laplace. If a is finite, the equation is linear in the second derivatives of the potential, but the coefficients are functions of the velocity components and the velocity of sound, which itself is a function of the magnitude of the resulting velocity according to the relation (6):

$$a^2 = a_0^2 - [(\gamma - 1)/2]v^2 \qquad (26)$$

Eq. (25) applied to the two-dimensional problem, with the notations $x_1 = x$, $x_2 = y$, $v_1 = v_x$, $v_2 = v_y$, can be written in the form

$$\left(1 - \frac{v_x^2}{a^2}\right)\frac{\partial^2\varphi}{\partial x^2} - 2\frac{v_x v_y}{a^2}\frac{\partial^2\varphi}{\partial x\partial y} + \left(1 - \frac{v_y^2}{a^2}\right)\frac{\partial^2\varphi}{\partial y^2} = 0 \qquad (27)$$

It appears that—with the exception of some cases which have very simple symmetry conditions—it is extremely difficult to obtain exact solutions of the complicated non-linear differential Eqs. (25) or (27). In the following sections methods for approximate solution of these differential equations are discussed.

Expansion in Powers of the Mach Number

Assume that the velocity potential is expanded in a power series of powers of the square of the Mach number M in the form:

$$\varphi = U[\Phi_0 + \Phi_1 M^2 + \Phi_2 M^4 + \ldots.] \qquad (28)$$

where U is the velocity of the undisturbed flow, $M = U/a$, and $\Phi_0, \Phi_1, \ldots.$ are functions of x_1, x_2, x_3.

Substituting (28) in Eq. (25) one obtains for the coefficient of the zero approximation the equation

$$\Delta\Phi_0 = 0 \qquad (29)$$

i.e., the equation for incompressible fluids. To obtain the coefficient of the next approximation Φ_1 the right side of Eq. (25) has to be expanded in powers of M^2, taking into account the expression (26) for a^2. Then one obtains

$$\Delta\Phi_1 = \sum_{i=1}^{3}\sum_{k=1}^{3} \frac{\partial\Phi_0}{\partial x_i}\frac{\partial\Phi_0}{\partial x_k}\frac{\partial^2\Phi_0}{\partial x_i\partial x_k} \qquad (30)$$

In a more physical language, after solving the flow problem for incompressible fluids, the terms on the right side of Eq. (30) are being computed using this solution and are being substituted in the equation. Therefore Eq. (30) has the form

$$\Delta\Phi_1 = F(x_1, x_2, x_3) \qquad (31)$$

of a non-homogeneous linear differential equation. If Φ_1 is known, an equation of similar form is obtained for Φ_2, and so on for the higher coefficients.

The problem of solving Eq. (31) for the given boundary conditions at the surface of the body is identical with the problem of finding the flow of an incompressible fluid around the body produced by a field of continuously distributed sources of the intensity $F(x_1, x_2, x_3)$ outside of the body. The flow produced by a source around bodies of certain simple shapes like, for example, circular or elliptical cylinders, sphere or ellipsoids is known and therefore in such cases the solution of Eq. (30) can be obtained by superposition. However, the procedure is very laborious, especially for the higher coefficients Φ_2, Φ_3, etc., and if $M > 0.4$ the series (28) converges very slowly.

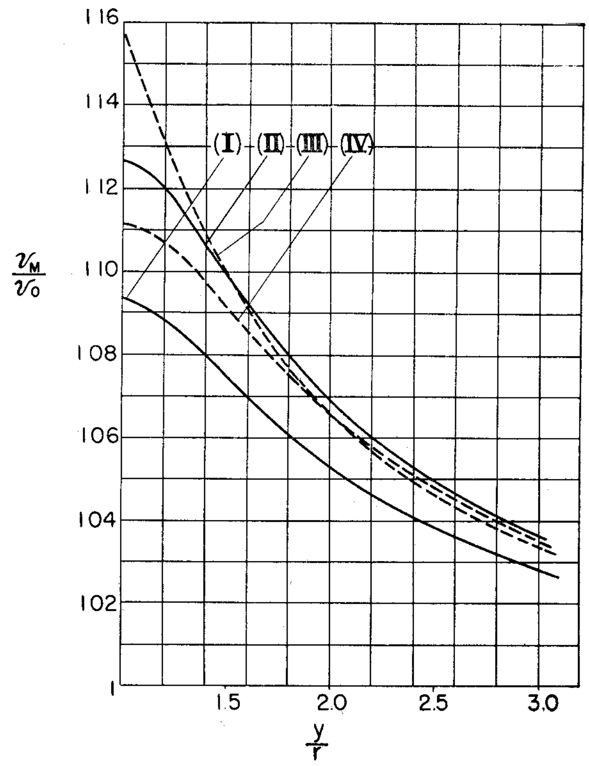

FIG. 5. The velocity correction factors v_M/v_0 for flow around a circular cylinder.
(I) First approximation.
(II) Second approximation, $\gamma = 1.405$.
(III) Hodograph method, $\gamma = -1$, distortion uncorrected.
(IV) Hodograph method, $\gamma = -1$, distortion corrected.

The method of expansion—which can be also considered as an iteration method—was used by Janzen and Lord Rayleigh for computation of the flow of a compressible fluid around a circular cylinder; then by I. Imai, and by K. Tamada and Y. Saito.

For the two-dimensional case Imai and Aihara gave an elegant method for finding a particular integral of the non-homogeneous equation for the first approximation:

$$\Delta\Phi_1 = \left(\frac{\partial\Phi_0}{\partial x}\right)^2\frac{\partial^2\Phi_0}{\partial x^2} + 2\frac{\partial\Phi_0}{\partial x}\frac{\partial\Phi_0}{\partial y}\frac{\partial^2\Phi_0}{\partial x\partial y} + \left(\frac{\partial\Phi_0}{\partial y}\right)^2\frac{\partial^2\Phi_0}{\partial y^2} \qquad (32)$$

It is known that if $\Phi_0(x, y)$ is a solution of Laplace's equation, Φ_0 is the real part of the complex function $F = \Phi_0 + i\Psi_0$ of the complex variable $z = x + iy$. Introduce the complex conjugate of F; namely, $\bar{F} = \Phi_0 - i\Psi_0$ and the conjugate of z; namely, $\bar{z} = x - iy$. Then the right side of Eq. (32) can be written in the simple form $R(d^2\bar{F}/d\bar{z}^2)(dF/dz)^2$ and it can be shown that the real part of the complex quantity $(d\bar{F}/d\bar{z})\int(dF/dz)^2dz$ is a particular integral of Eq. (32). To find the complete solution one has to add an integral of the homogeneous equation $\Delta\Phi_1 = 0$ which together with the particular integral obtained satisfies the boundary conditions.

It is seen that provided a closed-form solution of the flow problem for incompressible fluids is known, as, for example, in the case of a circular or elliptical cylinder or a Joukowsky airfoil, the first approximation for a compressible fluid can be obtained by direct integration and solution of Laplace's equation for given boundary values.

Fig. 5 refers to the flow around a circular cylinder. It gives the correction factor to be applied to the velocity distribution over a section passing through the center of the circle normal to the main flow direction; y is the distance from the center, r the radius of the cylinder. The correction factor plotted in the diagram is v_M/v_0, where v_M refers to $M = 0.4$ and v_0 to the solution for incompressible fluids. The two curves (I) and (II) show the first and second approximations obtained by Imai and Tamada by using the method of sources (developed by L. Poggi). The value $\gamma = 1.405$ is used for the second approximation; the first approximation is independent of the value of γ. The rest of the curves plotted in the figure will be discussed in the section on Applications of the Hodograph Method.

The following table shows the particular cases treated by means of the method of expansion up to the present.

	First Approximation	Second Approximation
Circular cylinder	Lord Rayleigh, O. Janzen, L. Poggi	I. Imai, K. Tamada and Y. Saito
Elliptic cylinder	S. G. Hooker, C. Kaplan, I. Imai and T. Aihara	
Joukowsky airfoil	L. Poggi, C. Kaplan	
Sphere	K. Tamada	K. Tamada

Method of Small Perturbations

It is known that the so-called thin airfoil theory has been applied with success to airfoils of small thickness. This theory is based on the assumption that the velocity changes caused in a uniform parallel airstream by the presence of the airfoil are small in comparison with the velocity of the undisturbed flow.

The same assumption was applied by Glauert, Ackeret and Prandtl to the flow of a compressible fluid around airfoils or slender bodies. Denote the velocity of the undisturbed flow by V and assume that the squares and products of the quantities $(v_1 - V)$, v_2 and v_3 can be neglected. Then Eq. (24) obtains the form

$$\sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i} = \frac{V^2}{a^2}\frac{\partial v_1}{\partial x_1} \tag{33}$$

The local velocity of sound is determined by the relation of Eq. (6).

$$a^2 + \frac{\gamma - 1}{2} v^2 = a_1^2 + \frac{\gamma - 1}{2} V^2 \tag{34}$$

It is seen that the difference between a^2 and a_1^2, *i.e.*, between the squares of the local velocity of sound and the velocity of sound in the undisturbed fluid can be neglected in this approximation. With the introduction of the velocity potential in the form $\varphi = Vx + \varphi'$ Eq. (33) furnishes the following linear differential equation for the "perturbation potential" φ':

$$\left(1 - \frac{V^2}{a_1^2}\right)\frac{\partial^2\varphi'}{\partial x_1^2} + \frac{\partial^2\varphi'}{\partial x_2^2} + \frac{\partial^2\varphi'}{\partial x_3^2} = 0 \tag{35}$$

This equation can be reduced to Laplace's equation provided $V < a_1$, by introducing as new independent variables $\xi_1 = x_1$, $\xi_2 = x_2\sqrt{1 - M_1^2}$ and $\xi_3 = x_3\sqrt{1 - M_1^2}$, where $M_1 = V/a_1$ is the Mach number for the undisturbed flow. If $(V/a_1) > 1$, *i.e.*, the undisturbed flow is supersonic, Eq. (35) has the character of the wave equation.

From Eq. (35) it is seen that the potential function for a flow in an incompressible fluid around a surface whose equation is given by $x_2 = f(x_1)$, $x_3 = g(x_1)$ can be expressed approximately in the form

$$\varphi(x_1, x_2, x_3) = Vx_1 + \varphi'(x_1, x_2\sqrt{1 - M_1^2}, x_3\sqrt{1 - M_1^2}) \tag{36}$$

where φ' is a solution of the Laplace equation for the variables $\xi_1 = x_1$, $\xi_2 = x_2\sqrt{1 - M_1^2}$, $\xi_3 = x_3\sqrt{1 - M_1^2}$. This result can be interpreted in the following way.

Evidently

$$\varphi(\xi_1, \xi_2, \xi_3) = V\xi_1 + \varphi'(\xi_1, \xi_2, \xi_3)$$

represents a flow of an incompressible fluid in the ξ_1, ξ_2, ξ_3 coordinate system. One has then to investigate the shape of the body in this system by comparison of the streamlines in the x_1, x_2, x_3 and the ξ_1, ξ_2, ξ_3 coordinate system. For sake of simplicity take a body with axial symmetry and compute the inclination β of an arbitrary streamline in the $x_1\, x_2$ plane. With the approximation used in this section $\tan\beta = (1/V)(\partial\varphi'/\partial x_2)$ in the x_1, x_2, x_3 system and $\tan\beta' = (1/V)(\partial\varphi'/\partial\xi_2)$ in the ξ_1, ξ_2, ξ_3 system. It is seen that

$$\frac{\tan\beta'}{\tan\beta} = \frac{\partial\varphi'}{\partial\xi_2}\bigg/\frac{\partial\varphi'}{\partial x_2} = \frac{1}{\sqrt{1 - M_1^2}} \tag{37}$$

From Eq. (37) it follows that the flow in the ξ_1, ξ_2, ξ_3 system represents the flow around a body obtained from the original body by expanding its lateral dimension in the ratio $1:\sqrt{1 - M_1^2}$.

Therefore, if one has to determine the flow of a compressible fluid around a slender body, the Mach number of the undisturbed flow being equal to M_1, this can be accomplished approximately by expanding the dimensions of the body normal to the main flow in the ratio $1:\sqrt{1 - M_1^2}$ and solving the problem of the flow of an incompressible fluid around the body of the expanded shape. Especially it can be shown that with the approximation used in this section the pressures acting on corresponding points of the two bodies are equal.

Namely neglecting the squares and products of perturbance velocity components considered small in comparison with V the pressure at an arbitrary point of a compressible or incompressible fluid is equal to

$$p - p_1 = -\rho_1 V(v_1 - V) = -\rho_1 V v_1' \qquad (38)$$

where v_1' denotes the perturbation velocity in the direction of the main flow and p_1 refers to the undisturbed flow. Since $(\partial\varphi'/\partial x_2) = (\partial\varphi'/\partial\xi_2)$, *i.e.*, the v_1' components are equal in the two cases, the pressures are also approximately equal.

Compare now the pressure distribution acting in an incompressible fluid on two slender bodies, assuming that the shape of the second is obtained by lateral expansion in the ratio $1:n$. If the potential function for the flow around the first body is

$$\varphi = V\xi_1 + \varphi'(\xi_1, \xi_2, \xi_3)$$

the potential function for the flow around the second body is approximately given by

$$\varphi = V\xi_1 + n\varphi'(\xi_1, \xi_2, \xi_3) \qquad (39)$$

This can be shown by consideration of the inclination of the streamlines in the manner indicated above. Then it is seen from Eq. (38) that the perturbation velocities v_1' and the pressure differences $p - p_1$ are also in the ratio $1:n$.

This result can be applied to an approximate calculation of the pressure coefficient in a compressible fluid if the value of the pressure coefficient for the flow of an incompressible fluid around the same body is known. It was shown that the pressure acting in a compressible fluid is equal to the pressure acting in an incompressible fluid on a body obtained by expansion in the ratio $1:\sqrt{1 - M_1^2}$. The incompressible flow around this body can now be compared with the incompressible flow around the original body. As it was explained above, the pressure coefficients vary proportionally with the lateral dimensions; hence the pressure coefficients at corresponding points are in the ratio $1:\sqrt{1 - M_1^2}$. Therefore, if the pressure coefficient at an arbitrary point of a body in an incompressible fluid is equal to $C_{p_0} = \Delta p/(\rho/2)V^2$, the pressure coefficient in a compressible fluid is approximately equal to $C_{p_M} = (1/\sqrt{1 - M_1^2})C_{p_0}$.

This relation of course cannot be correct for the neighborhood of the stagnation point, where the perturbance velocity is of the same order as the undisturbed velocity V. Nevertheless, if the influence of such regions is small, the same rule applies to the total forces acting on the body. For example, as was shown by Glauert, Ackeret and Prandtl, the lift coefficient of an airfoil moving with high velocity can be obtained from the lift coefficient obtained for small values of M_1 by the formula

$$C_{L_M} = (1/\sqrt{1 - M_1^2})C_{L_0} \qquad (40)$$

In the case of an airfoil the "expansion" normal to the main flow is essentially equivalent to a change of angle of attack, camber and thickness.

According to Eq. (40) the slope of the lift curve $dC_L/d\alpha$ should also vary in proportion to $1/\sqrt{1 - M_1^2}$. However, the comparison with experimental results does not permit one to formulate convincing final conclusions. In some cases good agreement is found; in other cases the rule is at variance with the experimental evidence. It is the author's impression that in most cases the Glauert-Prandtl rule underestimates the compressibility effect on the slope of the lift curve. For example, in Fig. 6 the ratio $(dC_L/d\alpha)_M/(dC_L/d\alpha)_0$ is plotted as a function of M_1. The experimental points

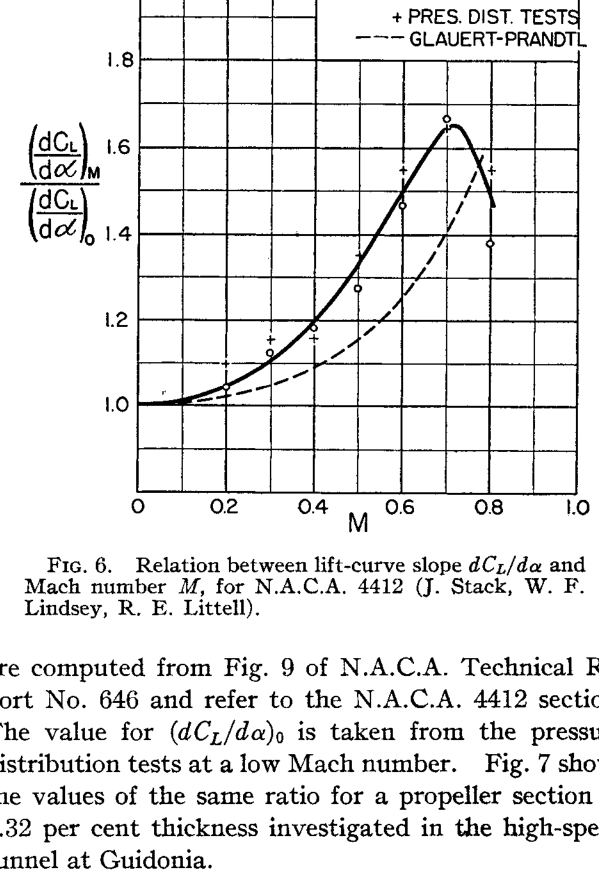

FIG. 6. Relation between lift-curve slope $dC_L/d\alpha$ and Mach number M, for N.A.C.A. 4412 (J. Stack, W. F. Lindsey, R. E. Littell).

are computed from Fig. 9 of N.A.C.A. Technical Report No. 646 and refer to the N.A.C.A. 4412 section. The value for $(dC_L/d\alpha)_0$ is taken from the pressure distribution tests at a low Mach number. Fig. 7 shows the values of the same ratio for a propeller section of 6.32 per cent thickness investigated in the high-speed tunnel at Guidonia.

The slope $dC_L/d\alpha$ reaches a peak value at a Mach number between about 0.6 and 0.8; beyond this peak a more or less rapid drop takes place. This drop apparently coincides with the first occurrence of a shock wave. This effect will be discussed in a later section.

The Hodograph Method

The analytical method presented in this section is restricted to two-dimensional problems. In the section on One-Dimensional Gasdynamics the fundamental differential equation for the two-dimensional irrotational motion of a compressible fluid in Cartesian coordinates is given. In many problems it is convenient to use the arc length along streamlines, and equipotential lines as

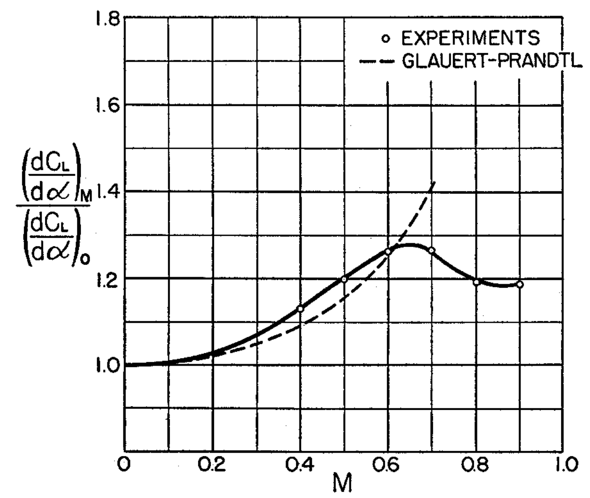

FIG. 7. Relation between lift-curve slope $dC_L/d\alpha$ and Mach number M, for a 6 per cent thick Italian propeller section (A. Ferri).

independent variables; the magnitude of the velocity v and the angle of inclination between the velocity vector and a fixed direction (*e.g.*, the direction of the undisturbed flow) as unknown functions. Then the velocity

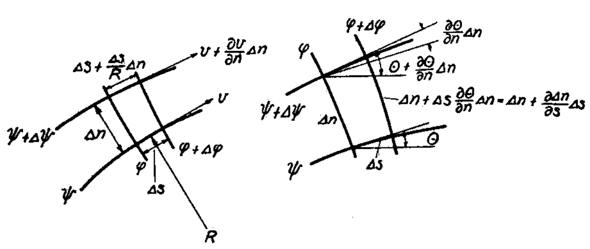

FIG. 8.

distribution is determined by the conditions of irrotationality and continuity. The rotation of the fluid in the case of curved flow is given by (Fig. 8)

$$\omega = (\partial v/\partial n) + (v/R) \tag{41}$$

where v is the magnitude of the velocity, n is the arc length taken along an equipotential line, and R is the radius of curvature of the streamline. Evidently $1/R = \partial\Theta/\partial s$, where s is the arc length taken along a streamline. Hence, one obtains as the condition of irrotationality:

$$\frac{1}{v}\frac{\partial v}{\partial n} - \frac{\partial \Theta}{\partial s} = 0 \tag{42}$$

The continuity condition requires that for an elementary channel between two neighboring streamlines $\rho v \Delta n = $ const., *i.e.*,

$$\frac{1}{\rho}\frac{\partial \rho}{\partial s} + \frac{1}{v}\frac{\partial v}{\partial s} + \frac{1}{\Delta n}\frac{\partial \Delta n}{\partial s} = 0 \tag{43}$$

According to Eq. (19)

$$\frac{1}{\rho}\frac{\partial \rho}{\partial s} + \frac{1}{v}\frac{\partial v}{\partial s} = (1 - M^2)\frac{1}{v}\frac{\partial v}{\partial s} \tag{44}$$

On the other hand, it is seen from Fig. 8 that

$$\frac{1}{\Delta n}\frac{\partial \Delta n}{\partial s} = \frac{\partial \Theta}{\partial n} \tag{45}$$

Hence one obtains the equation

$$(1 - M^2)\frac{1}{v}\frac{\partial v}{\partial s} + \frac{\partial \Theta}{\partial n} = 0 \tag{46}$$

The two Eqs. (42) and (46) determine the velocity distribution in the following sense. Assume one draws arbitrarily an orthogonal system of curves $\varphi = $ const., $\psi = $ const. Then by integrating Eqs. (42) and (46), for example, by a step by step method, one obtains certain values for v and Θ. Obviously the values of the angle Θ obtained in this way should be consistent with the arbitrarily assumed streamlines, Θ indicating the direction of the velocity vector and therefore the streamline at an arbitrary point. If this is not the case, one has to correct the network and continue the procedure which is essentially a trial and error method. The Russian mathematician, Chaplygin, noticed that the problem can be greatly simplified if a change of variables is performed by using v and Θ as independent variables and φ and ψ as unknown functions. His method amounts essentially to plotting a network $\varphi = $ const., $\psi = $ const. in a plane in which the velocity components v_x and v_y are the Cartesian coordinates or v and Θ the polar coordinates. In other words, as one proceeds along a streamline or an equipotential line in the "physical plane" (the plane in which the flow actually takes place), one plots (Fig. 9) the velocity vector as to magnitude and direction starting from the origin in a second

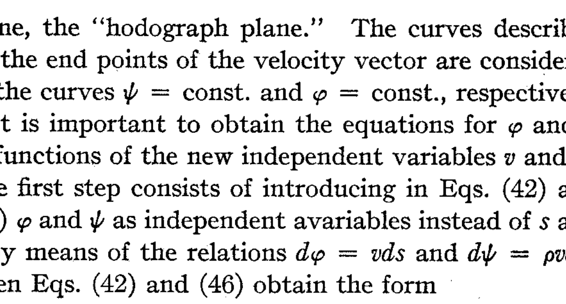

FIG. 9.

plane, the "hodograph plane." The curves described by the end points of the velocity vector are considered as the curves $\psi = $ const. and $\varphi = $ const., respectively.

It is important to obtain the equations for φ and ψ as functions of the new independent variables v and Θ. The first step consists of introducing in Eqs. (42) and (46) φ and ψ as independent avariables instead of s and n by means of the relations $d\varphi = vds$ and $d\psi = \rho vdn$. Then Eqs. (42) and (46) obtain the form

$$\frac{\rho}{v}\frac{\partial v}{\partial \psi} - \frac{\partial \Theta}{\partial \varphi} = 0$$

$$(1 - M^2)\frac{1}{v}\frac{\partial v}{\partial \varphi} + \rho\frac{\partial \Theta}{\partial \psi} = 0 \tag{47}$$

Now the independent and dependent variables have to be interchanged. This is a well-known operation in mathematical analysis. If y is a function of the independent variable x and one chooses to consider x as a function of y, obviously from $dx/dx = (dx/dy)(dy/dx) = 1$ follows $dx/dy = 1/(dy/dx)$. Using the obvious identities: $\partial\varphi/\partial\varphi = \partial\psi/\partial\psi = 1$, $\partial\varphi/\partial\psi = \partial\psi/\partial\varphi = 0$, the following analogous relations between the partial differential coefficients are obtained:

$$\frac{\partial v}{\partial \psi} = \frac{1}{\Delta}\frac{\partial \varphi}{\partial \Theta}, \quad \frac{\partial \Theta}{\partial \psi} = -\frac{1}{\Delta}\frac{\partial \varphi}{\partial v}$$

$$\frac{\partial v}{\partial \varphi} = -\frac{1}{\Delta}\frac{\partial \psi}{\partial \Theta}, \quad \frac{\partial \Theta}{\partial \varphi} = \frac{1}{\Delta}\frac{\partial \psi}{\partial v} \tag{48}$$

where Δ is the so-called "Jacobian"; defined by

$$\Delta = \frac{\partial \varphi}{\partial v}\frac{\partial \psi}{\partial \Theta} - \frac{\partial \varphi}{\partial \Theta}\frac{\partial \psi}{\partial v} \tag{49}$$

The expression Δ is denoted in general by $\partial(\varphi, \psi)/\partial(v, \Theta)$ and has to some extent properties analogous to a differential quotient. For example, the product of differentials $d\varphi d\psi$ is equal to $(\partial(\varphi, \psi)/\partial(v, \Theta))dvd\Theta$, just as the differential dy is equal to $(dy/dx)dx$.

Introducing the expression (48) in Eq. (47) one obtains:

$$\frac{\rho}{v}\frac{\partial \varphi}{\partial \Theta} - \frac{\partial \psi}{\partial v} = 0$$

$$(1 - M^2)\frac{1}{\rho v}\frac{\partial \psi}{\partial \Theta} + \frac{\partial \varphi}{\partial v} = 0 \tag{50}$$

Eliminating φ the calculation results in obtaining the following linear partial differential equation for ψ as a function of v and Θ:

$$\frac{1 - M^2}{\rho^2}\frac{\partial^2 \psi}{\partial \Theta^2} + \left(\frac{v}{\rho}\frac{\partial}{\partial v}\right)^2 \psi = 0 \tag{51}$$

It is evident that since v and Θ are the independent variables, and ρ and M are functions of v, this equation is a linear differential equation with variable coefficients, but the variable coefficients are functions of the independent variables only whereas in the case of Eq. (27) the coefficients are functions of the derivative of the unknown variable. Therefore, the solution of the Eq. (27) can be obtained only by a laborious iteration process, whereas the methods of analysis can be applied to Eq. (51) for obtaining families of exact solutions. This does not mean that one is able to solve exactly the flow problem for given boundary conditions in the physical plane, but one obtains exact solutions in the hodograph plane, which can be transferred to the physical plane and furnish certain exact flow patterns of a compressible fluid.

Chaplygin and his followers noticed that Eq. (51) is reduced to Laplace's equation if it is assumed that $(1 - M^2)/\rho^2$ is a constant quantity. Using the general thermodynamic relations, Eqs. (6) and (7), this expression is obtained in the form

$$\frac{1 - M^2}{\rho^2} = \frac{1}{\rho_0^2}\left(1 - \frac{\gamma + 1}{2}\frac{v^2}{a_0^2}\right) \times \left(1 - \frac{\gamma - 1}{2}\frac{v^2}{a_0^2}\right)^{(1+\gamma)/(1-\gamma)} \tag{52}$$

It is seen that $1 - M^2/\rho^2 = \text{const.}$ if $\gamma = -1$. For this value of γ the relation between density and velocity is given by Eq. (7)

$$\rho = \rho_0[1 + (v^2/a_0^2)]^{-1/2} \tag{53}$$

and according to Eq. (6) the square of the velocity of sound is equal to

$$a^2 = a_0^2[1 + (v^2/a^2)] = a_0^2\,(\rho_0^2/\rho^2) \tag{54}$$

Now using the definition of a^2

$$a^2 = dp/d\rho = a_0^2(\rho_0^2/\rho^2) \tag{55}$$

one obtains by integration the pressure-density relation

$$p = \text{const.} - (a_0^2\rho_0^2/\rho) \tag{56}$$

This relation, of course, is not satisfied by any real gas, since for all real gases $1 < \gamma < 1.66$, however, the constant of integration in Eq. (53) can be determined in such a way that in the p, $1/\rho$ diagram (Fig. 10) the

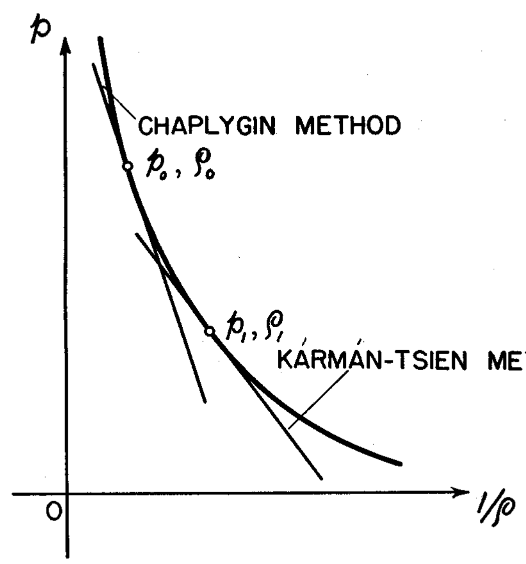

FIG. 10. Approximations to true isentropic pressure-volume curve.

straight line defined by Eq. (53) is tangent to the actual isentropic curve of the real gas. Chaplygin determined the constant so that the straight line approximates the isentropic curve near the point corresponding to the state of the gas in rest. In this way he and his followers solved certain problems by integration of Eq. (51) and obtained good approximations if the maximum velocity did not exceed about 0.4 times the velocity of sound.

In order to apply the hodograph method to larger velocities one can determine the constant of integration in Eq. (55) so that the isentropic curve of the gas is approximated near the point p_1, $1/\rho_1$ which corresponds to the undisturbed flow with velocity V. This idea was carried out at the suggestion of the present author by H. S. Tsien, and will be discussed in the following section.

Applications of the Hodograph Method

If the quantity $(1 - M^2)/\rho^2$ is assumed constant, after division of Eq. (51) by $(1 - M^2)/\rho^2$, this constant factor can be included in the bracket and one obtains

$$\frac{\partial^2\psi}{\partial\Theta^2} + \left(\frac{v}{\sqrt{1 - M^2}}\frac{\partial}{\partial v}\right)^2 \psi = 0 \qquad (57)$$

Assume that w, Θ denote the magnitude and the angle of inclination of the velocity vector of an incompressible fluid. Then for an arbitrary flow pattern of this incompressible fluid the stream function satisfies the equation

$$\partial^2\psi/\partial\Theta^2 + [w(\partial/\partial w)]^2\psi = 0 \qquad (58)$$

Consequently, if $\psi(w, \Theta)$ is a known flow pattern of an incompressible fluid and w and v are connected by the relation

$$\sqrt{1 - M^2}(dv/v) = (dw/w) \qquad (59)$$

the function $\psi(w(v), \Theta)$ determines the flow of a compressible fluid characterized by the pressure-density ratio (56). The factor $\sqrt{1 - M^2}$ can be expressed by the local velocity v, the velocity V and the Mach number M_1 of the undisturbed flow. Taking into account the relation

$$a^2 - v^2 = a_1^2 - V^2$$

where a_1 is the velocity of sound in the undisturbed flow, one obtains

$$M^2 = \frac{v^2}{a^2} = \frac{v^2}{a_1^2 - V^2 + v^2} = \frac{M_1^2(v/V)^2}{1 - M_1^2 + M_1^2(v/V)^2}$$

and

$$\sqrt{1 - M^2} = \sqrt{1 - M_1^2}/\sqrt{1 - M_1^2 + M_1^2(v/V)^2}$$

Hence, the relation between the velocities of the incompressible and compressible fluids is given by

$$dw/w = \sqrt{1 - M_1^2}/\sqrt{1 - M_1^2 + M_1^2(v/V)^2}(dv/v) \qquad (60)$$

By integrating and putting $w\ (V) = W$ one obtains

$$\frac{w}{W} = \frac{v}{V}\frac{1 + \sqrt{1 - M_1^2}}{\sqrt{1 - M_1^2 + M_1^2(v/V)^2} + \sqrt{1 - M_1^2}}$$

or

$$v/V = (w/W)(1 - \lambda)/1 - \lambda(w/W)^2 \qquad (61)$$

where

$$\lambda = M_1^2/(1 + \sqrt{1 - M_1^2})^2$$

The pressure coefficient corresponding to the Mach number $M_1 = V/a_1$ is given by

$$C_{p_M} = \frac{2(p - p_1)}{\rho_1 V_1^2} =$$

$$C_{p_0}\frac{1}{\sqrt{1 - M_1^2} + \dfrac{M_1^2}{1 + \sqrt{1 - M_1^2}}\dfrac{C_{p_0}}{2}} \qquad (62)$$

provided C_{p_0} is the value of the pressure coefficient for $M_1 = 0$. It is seen that Eq. (62) differs from the Glauert-Prandtl formula by the second term in the denominator which itself contains C_{p_0} again. Hence, if C_{p_0} is negative, the magnitude of the compressibility effect predicted by Eq. (62) is greater than that predicted by the Glauert-Prandtl formula. The formula (62) can be applied to calculation of the influence of compressibility on the pressure distribution of airfoils, if the pressure distribution at low Mach numbers is known. In this application a slight distortion of the shape of the airfoil is disregarded; namely, if $\psi(w, \Theta)$ is the stream function for the flow of an incompressible fluid around a given section, so that, for example, the streamline $\psi = 0$ corresponds to the boundary of the section, then the streamline $\psi = 0$ of the corresponding compressible flow in the physical plane will not coincide exactly with the boundary of the original section. It appears—at least in an example investigated by Tsien and discussed later in this section—that the distortion of the section is relatively small (with the exception of such thick sections as a circle) and the error made in the pressure coefficient tends to correct the error made by use of the approximate pressure-density relation.

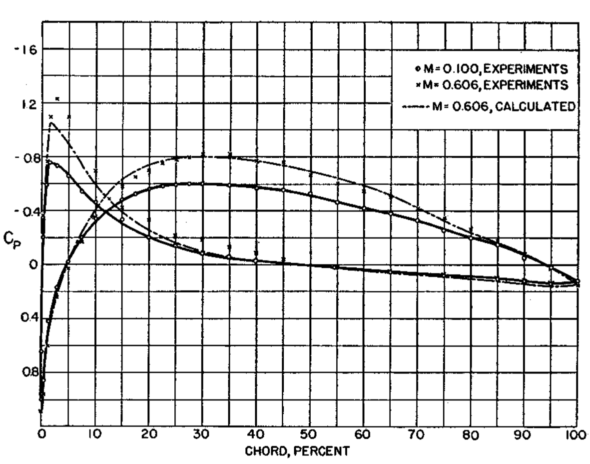

Fig. 11. Pressure distributions for N.A.C.A. 4412 at $\alpha = -2°$.

In Fig. 11 the measured pressure distributions of the N.A.C.A. 4412 airfoil are plotted for $M_1 = 0.100$ and $M_1 = 0.606$ according to N.A.C.A. Technical Report No. 646. The dotted line is calculated for $M_1 = 0.606$

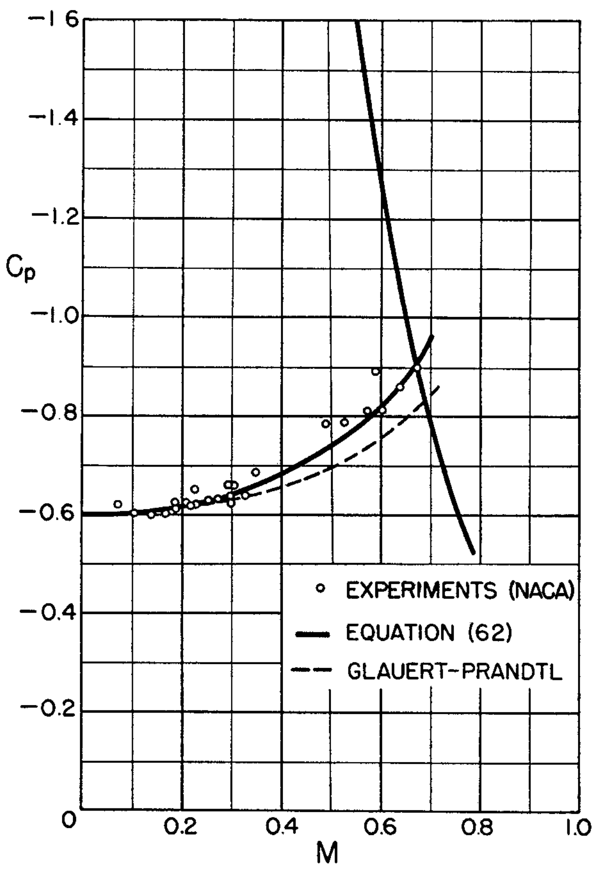

FIG. 12 (a). Increase of the pressure coefficient C_p with Mach number M, for N.A.C.A. 4412. At 30 per cent chord, upper surface, $\alpha = -2°$.

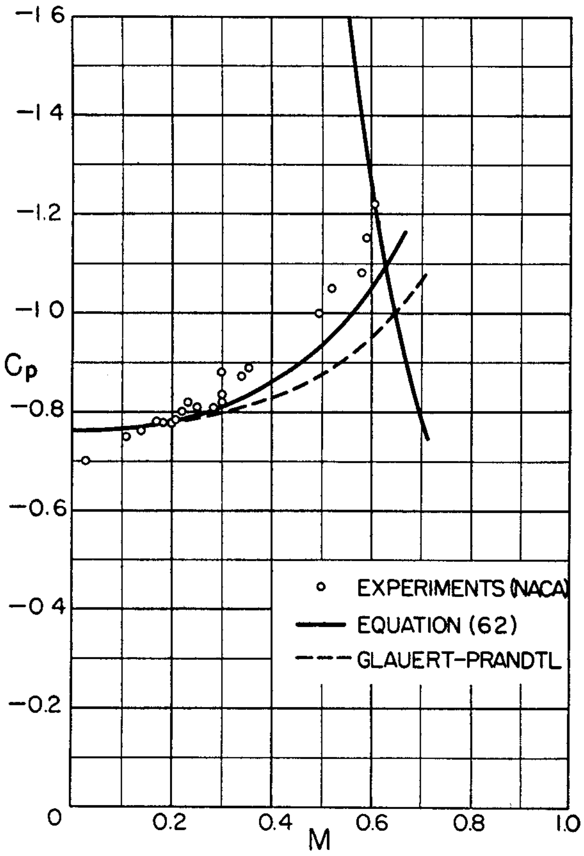

FIG. 12 (b). Increase of the pressure coefficient C_p with Mach number M, for N.A.C.A. 4412. At 27.5 per cent chord, upper surface, $\alpha = -0°15'$.

by Eq. (62) using for C_{p_0} the values measured at $M_1 = 0.100$ and $\alpha = -2$ degrees.

Figs. 12 (a, b, c) show the variation of pressure coefficients at three points of the same airfoil section; the measured values are compared with the values computed by Eq. (62) and by the Glauert-Prandtl rule.

Strictly speaking, the theory developed in this section deals with a specific kind of compressible fluid for which $\gamma = -1$. However, one can look at the method also from a different point of view. For air, the factor $(1-M^2)/\rho^2$ in Eq. (51) is variable, but a fair approximation might be expected if this variable coefficient is replaced by its mean value. This is probably correct unless the local Mach number comes close to unity. Replacing $(1 - M^2)/\rho^2$ by its value corresponding to the undisturbed fluid, *i.e.*, by $(1 - M_1^2)/\rho_1^2$, one obtains the following equation for ψ:

$$\frac{\partial^2\psi}{\partial\Theta^2} + \left(\frac{v}{\rho}\frac{\rho_1}{\sqrt{1 - M_1^2}}\frac{\partial}{\partial v}\right)^2 \psi = 0 \tag{63}$$

Then Eq. (59) is replaced by

$$\sqrt{1 - M_1^2}\,\frac{\rho}{\rho_1}\frac{dv}{v} = \frac{dw}{w} \tag{64}$$

Eq. (64) leads to the following approximate relation between v and w:

$$(w/W)^k = (v/V)e^{-0.230(V/a_0)^2[(v^2/V^2)-1]} \tag{65}$$

where

$$k = \left(1 + \frac{\gamma - 1}{2} M_1\right)^{-1/(\gamma-1)} / \sqrt{1 - M_1^2}$$

The pressure coefficients calculated from (65) are shown in Fig. 13; they are only slightly different from those obtained from Eq. (62) plotted in the same figure by dotted lines. For larger values of M_1 Eq. (65) apparently gives better agreement with the experiments.

The physical meaning of the Eqs. (63) and (57) can be described in the following way. The stream function $\psi[w(v), \Theta]$ which is a solution of Eq. (63) furnishes a family of approximate streamlines for a compressible fluid satisfying the exact relations between pressure, density and velocity; whereas the stream function satisfying Eq. (57) furnishes exact streamlines of a compressible fluid whose physical behavior is only approximately correct. The second method has the advantage of simplicity in the mathematical expressions.

In order to have an estimate of the error made by use of the approximate pressure-density relation, a calculation was made in an extreme case; namely, in the case of a circular cylinder. Fig. 5, which was partly discussed before in the section on Expansion in Powers on the Mach Number, shows the velocity correction factors

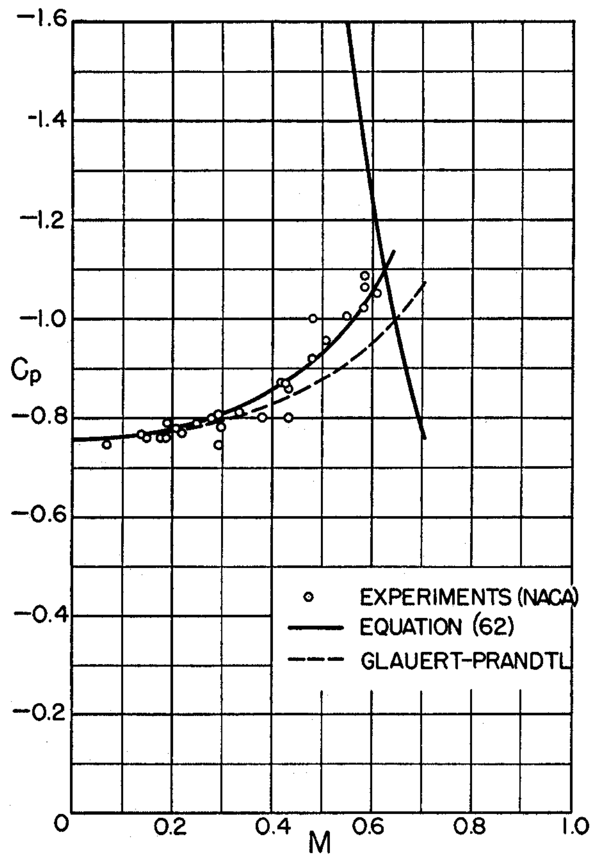

FIG. 12 (c). Increase of the pressure coefficient C_p with Mach number M, for N.A.C.A. 4412.
At 2.5 per cent chord, lower surface, $\alpha = -2°$.

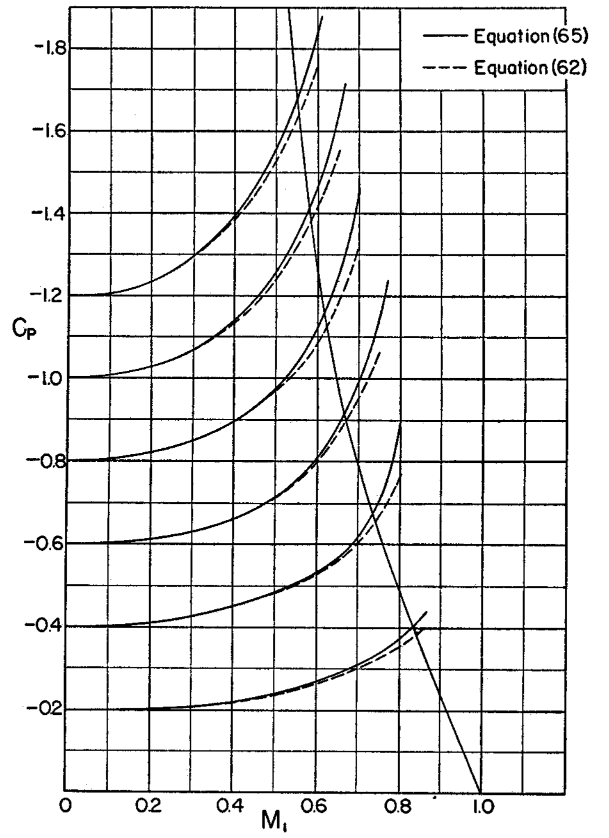

FIG. 13. Pressure coefficients C_p *vs.* Mach number calculated from Eqs. (62) and (65).

v_M/v_0 for $M_1 = 0.4$ calculated under two different assumptions. Curve III is obtained without correction of the distortion of the original section, Curve IV by solving the flow pattern for an originally elliptical cylinder which by the distortion becomes almost exactly circular (within $^1/_{50}$ of a per cent of the radius). In both cases the approximate pressure-density relation (56) is employed, which makes an exact solution of the flow equations possible. These curves can be compared with the second approximation obtained for $\gamma = 1.405$ (Curve II). This curve probably lies somewhat below the curve which would represent the exact solution for $\gamma = 1.405$. It is seen that the error made by the distortion lies in the right direction to balance the error made by using $\gamma = -1$ instead of the correct value of γ.

Drag and Transition through the Velocity of Sound

The experimental evidence shows that the compressibility effect on the drag of airfoil sections can be attributed to two different physical processes. At low values of the Mach number the drag of most airfoil sections consists essentially of skin friction of partly laminar and to a greater part turbulent character. The compressibility effect on the skin friction coefficient of a boundary layer with a fixed pressure gradient is very small in the range of the Mach numbers considered. However, with increasing Mach number the local velocity at the surface increases faster than in the undisturbed stream and this causes an increase of the skin friction. Another effect is due to the increase of adverse pressure gradient at the rear portion of the airfoil causing an increase in thickness of the boundary layer. The increase of the adverse pressure gradient is caused by the large increase of the maximum suction shown in the sections on Method of Small Perturbations and Applications of the Hodograph Method. Both effects can be taken into account—at least approximately—by the theoretical considerations given in the section on Method of Small Perturbations. It was pointed out that the velocity and pressure distribution along the surface of a slender body or an airfoil at a certain Mach number is approximately the same as in a flow of an incompressible fluid provided the body is expanded laterally in the ratio $1:\sqrt{1 - M_1^2}$. This reasoning furnishes a method for estimating the compressibility effect on the drag in the following way. Consider first, symmetrical sections. If the thickness ratio of the airfoil is denoted by t, the fictitious thickness corresponding to the Mach number M_1 is equal to $t/\sqrt{1 - M_1^2}$. Provided the influence of the thickness on the drag is known for the family of airfoils considered, the minimum drag coefficient C_{D_0} can be put approximately equal to the minimum drag coefficient C_{D_0} of a section of the thickness $t/\sqrt{1 - M_1^2}$. To estimate the correction for an arbitrary angle of attack it has to be taken into account that by the lateral expansion of the flow pattern the angle of attack is also increased in the ratio $1/\sqrt{1 - M_1^2}$. Therefore, the drag coefficient of a section of thickness t at the angle of attack α and Mach number M_1 will be approximately equal to the drag coefficient of a section of the thickness $t/\sqrt{1 - M_1^2}$ at the angle of attack $\alpha/\sqrt{1 - M_1^2}$. If the effect of compressibility on the lift coefficient is known, one can use the drag coefficient of the expanded section at the lift coefficient C_{L_M}, which corresponds to the Mach number M_1 at the angle of attack α. For cambered sections the camber has to be also increased in the ratio $1:\sqrt{1 - M_1^2}$.

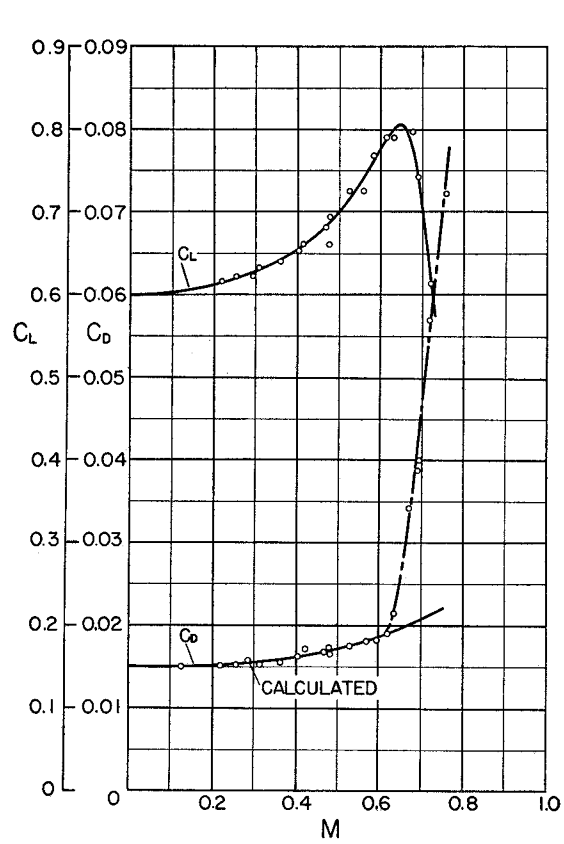

FIG. 14. Lift coefficient C_L and drag coefficient C_D for N.A.C.A. 4412 at $\alpha = 1°52'$.

Fig. 14 gives a comparison between values of the drag coefficient of the section N.A.C.A. 4412 measured at 1°52′ angle of attack and computed according to the method explained above. The influence of thickness was evaluated according to N.A.C.A. Technical Report No. 669 which apparently represents the last word in this somewhat controversial question. It was assumed that the relative influence of the thickness is independent of Reynolds number. It is seen that the accordance between the measured and estimated values is very good, until the "compressibility burble" occurs, indicated by simultaneous sudden drop of the lift coefficient and increase of the drag coefficient.

The "compressibility burble" is apparently caused by a shock wave extending from the surface to a certain distance from the surface. Since the magnitude of the pressure or the velocity jump is variable along the discontinuity surface, the flow behind the shock wave is a flow with continuously distributed vorticity even if the flow is irrotational in front of the discontinuity surface. Since far behind the airfoil as the flow becomes parallel again—at least theoretically—the shock wave must cause a non-uniform velocity distribution; *i.e.*, a wake in addition to the wake caused by the boundary layer. This has been experimentally shown by J. Stack, W. F. Lindsey, R. E. Littell and also by C. N. H. Lock.

There are only a few measurements available for drag and lift of airfoils beyond the velocity of sound and it seems that nobody has been able yet to obtain reliable measurements at the speed equal to the velocity of sound. The observations of the N.A.C.A. are extended essentially to the speed at which the compressibility burble occurs. The experimenters working with the high-speed tunnel in Guidonia found by increasing the speed beyond this limit a "zone of oscillation" in which no stable equilibrium can be established between the air forces and the balance. Beyond this zone the drag coefficient has a higher value and is apparently constant in the range of the velocity of sound. Fig. 15 shows the Italian measurements for an airfoil with 6.32 per cent thickness whose shape is almost identical with that of a Clark Y section. Curiously enough the N.A.C.A. measurements on a Clark Y airfoil of 6 per

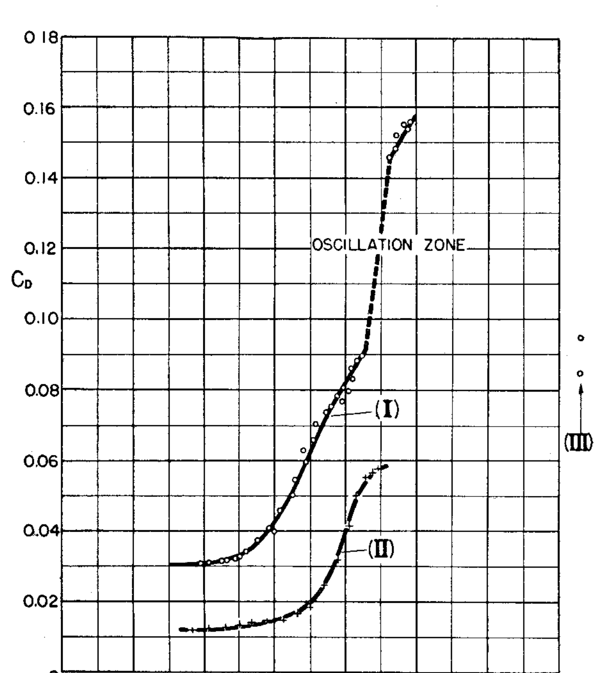

FIG. 15. Drag coefficient C_D *vs.* Mach number M, at $C_{L_0} \approx 0.45$.
(I) Italian propeller section of 6.32 per cent thickness (A. Ferri).
(II) Clark Y section of 6 per cent thickness (J. Stack).
(III) Göttingen 622 section of 8.85 per cent thickness for $\alpha = 2°$ and $4°$ at $M_1 = 1.47$ (A. Busemann and O. Walchner).

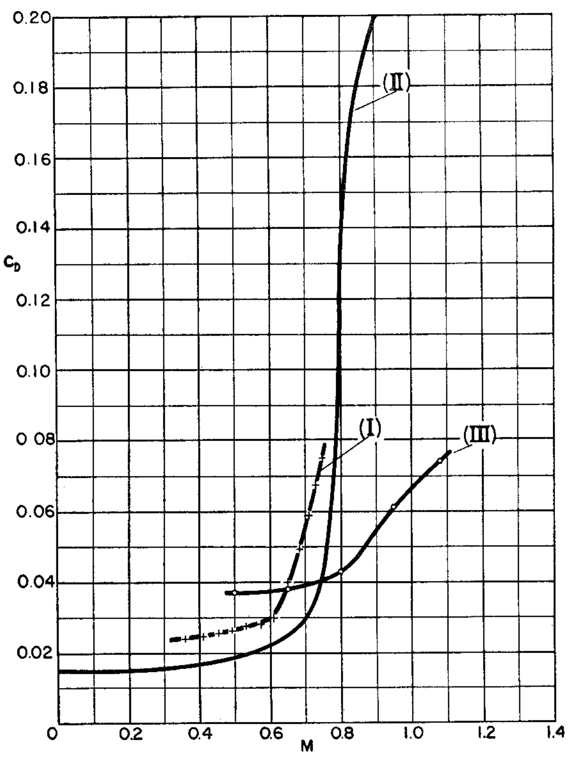

FIG. 16. Drag coefficient C_D of R.A.F. 6 section, 10 per cent thickness, at $\alpha = 2°$.
(I) N.A.C.A. tests (J. Stack).
(II) British propeller tests (G. P. Douglas).
(III) Open jet tests (L. J. Briggs and H. L. Dryden).

cent thickness are at wide variance with the Guidonia results. The two curves correspond to equal values of $C_L \approx 0.45$ for $M_1 \to 0$. For comparison the drag coefficients of a similar airfoil of 8.85 per cent thickness at $M_1 = 1.47$ for 2° and 4° angle of attack are indicated. These data are taken from measurements of A. Busemann. The N.A.C.A. data and results of British propeller tests concerning the compressibility effect on the drag of a 10 per cent thick R.A.F. 6 section at 2° angle of attack are in fair accordance (Fig. 16). In the same diagram the data obtained by Briggs and Dryden in an open high-speed jet are also plotted. It appears that these values are probably too low.

There are some measurements on lift and drag of a section with circular-arc upper and flat lower surface between $M_1 = 1.47$ and 2.13. These data are plotted in Figs. 17 and 18. In order to complete the picture as far as possible the corresponding data for the N.A.C.A. 4409–34 section are plotted for $0.35 < M_1 < 0.84$. This section is somewhat similar, having its maximum thickness at 40 per cent of the chord (instead of 50 per cent) and an almost flat bottom. It seems that the general shape of the drag curve is similar to that of the drag curve of a sharp-nosed projectile. The lift apparently decreases greatly after the first occurrence of the compressibility burble. The rise shown by the last points needs further confirmation. The theory of lift and drag of airfoils for supersonic speed has been developed by J. Ackeret, A. Busemann and S. G. Hooker.

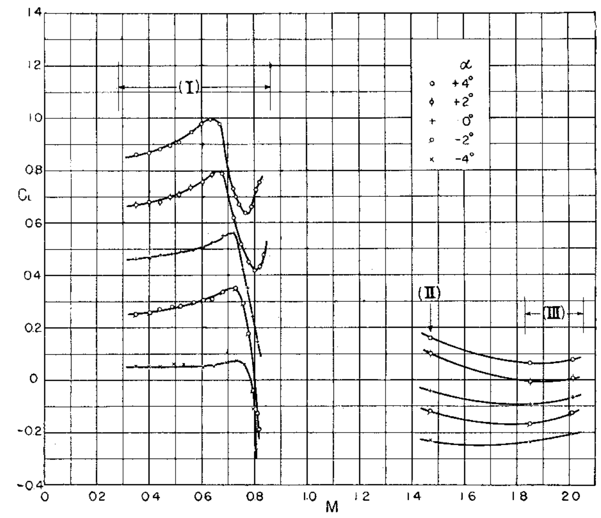

FIG. 17. Lift coefficient C_L *vs.* Mach number M.
(I) N.A.C.A. 4409–34 (J. Stack and A. E. v. Doenhoff).
(II) Circular-arc section of 8.85 per cent thickness (A. Busemann and O. Walchner).
(III) Circular-arc section of 8.8 per cent thickness (A. Ferri).

There are no data published on the compressibility effect on the drag of streamlined bodies such as fuselages and nacelles. In general it can be expected that if a certain streamlined shape has a favorable drag coefficient at low speed, it has to be expanded in the length direction or contracted laterally in a linear ratio about equal to $1:\sqrt{1 - M_1^2}$ in order to keep the drag coefficient low at high speed.

THE CRITICAL MACH NUMBER

The Mach number at which the compressibility

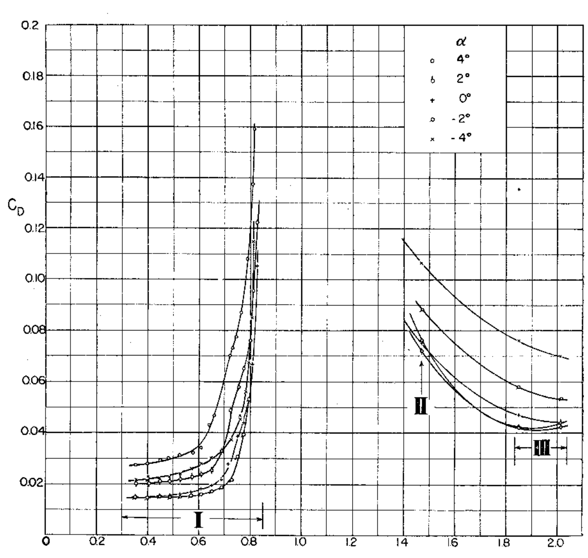

FIG. 18. Drag coefficient C_D *vs.* Mach number M.
(I) N.A.C.A. 4409–34 (J. Stack and A. E. v. Doenhoff).
(II) Circular-arc section of 8.85 per cent thickness (A. Busemann and O. Walchner).
(III) Circular-arc section of 8.8 per cent thickness (A. Ferri).

burble first occurs is called the critical Mach number. In general it is defined as the Mach number of the undisturbed flow for which the local velocity at some point of the surface reaches the local velocity of sound. Using Glauert's approximation the condition for the critical Mach number is given by

$$V + v_x' + a_1 \tag{66}$$

Assume that the pressure coefficient of a given airfoil or slender body at the point of maximum suction is equal to $-C_{p_0}$; then the corresponding pressure coefficient for the Mach number M_1 is

$$-C_{p_M} = -C_{p_0}/\sqrt{1 - M_1^2}$$

and since

$$C_{p_M} = 2v_x'/V$$

the Eq. (66) can be written in the form

$$V[1 + (C_{p_M}/2)] = a_1$$

or

$$M_1\left[1 + \left(C_{p_0}/2\sqrt{1 - M_1^2}\right)\right] = 1 \tag{67}$$

Hence, one obtains the following equation for the critical Mach number

$$2[(1 - M_1)^{3/2}(1 + M_1)^{1/2}]/M_1 = C_{p_0} \tag{68}$$

E. Jacobs, by using Eq. (8) for the relation between pressure and velocity, obtained the expression:

$$C_{p_0} = 2\sqrt{1 - M_1^2}\left[1 - \left(\frac{2 + (\gamma - 1)M_1^2}{\gamma + 1}\right)^{\gamma/(\gamma-1)}\right]\Big/\gamma M_1^2 \tag{69}$$

However, the question arises whether it is justified to introduce expressions involving higher terms in a linearized theory that neglects squares and products of the perturbation velocities. Possibly the error made in evaluation of the ratio C_{p_M}/C_{p_0} is of the same order as the correction involved by use of the exact thermodynamic relation (8) instead of (66).

The method developed by Tsien and the author permits the calculation of the critical Mach number by equating the pressure coefficient computed by Eq. (62) with the value of the pressure coefficient corresponding to a local Mach number $M = 1$ according to Eq. (8). In Figs. 10 to 12 this limit is plotted by a full line.

Fig. 19 shows a comparison of the relation "critical Mach number" *vs.* C_{p_0} according to the different approximate theories. For comparison a few experimental results are also indicated.

Of course, by laborious analytical methods it would be possible to refine the calculation of the critical Mach number, defined by the first coincidence between local velocity and local velocity of sound. However, it appears questionable whether this is the right criterion for the first appearance of a compressibility burble. A closer examination of the fundamental mathematical problem shows that at least in an ideal compressible fluid a shock wave does not necessarily occur if the local velocity exceeds the velocity of sound. The hodograph method can be used to throw some light on this question.

It was shown in the section on the Hodograph Method, that if $\psi(v, \Theta)$ is the stream function for a flow pattern of a compressible fluid where v is the magnitude of the velocity and Θ the angle of inclination of the velocity vector at an arbitrary point, then ψ as a function of v and Θ must satisfy the differential equation

$$\frac{1 - M^2}{\rho^2}\frac{\partial^2\psi}{\partial\Theta^2} + \left(\frac{v}{\rho}\frac{\partial}{\partial v}\right)^2\psi = 0 \tag{70}$$

Now it is possible to find exact particular integrals of this differential equation, and such solutions are supposed to give exact flow patterns of a compressible ideal fluid. The physicist or engineer is, of course, not satisfied by having determined streamlines in the hodograph plane and is anxious to see the flow pattern in the physical x, y plane. If the function $\psi(v, \Theta)$ is known, the computation of streamlines $\psi(x, y) = \text{const.}$ in the physical plane requires a further integration. However, in performing this integration it is found that in some cases a streamline that is perfectly continuous in the hodograph plane shows a singular behavior: plotting successive points of a streamline in the physical plane at a certain point which can be called the critical point, the streamline cannot be continued in the flow direction, and gives a quite impossible flow pattern with two families of crossing streamlines in the same plane. The velocity at this critical point is neither zero nor infinite. However, by continuing the calculation one obtains a second branch of the streamline, which meets with the first branch in a kind of "cusp." It appears that the flow cannot enter in a certain "forbidden"

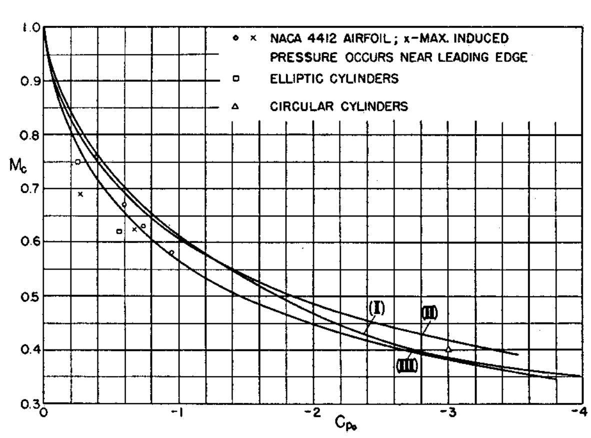

FIG. 19. Critical Mach number M_c *vs.* maximum suction-pressure coefficient at low speed C_{p_0}.
(I) Eq. (68).
(II) Eq. (69) (E. Jacobs).
(III) Hodograph method (Th. von Kármán and H. S. Tsien).

region without violating one of the physical assumptions underlying the analysis. The two assumptions are: continuity and irrotationality. Since the continuity cannot be violated, it must be assumed that the flow becomes rotational. In an ideal compressible fluid this can happen only through a pressure jump; *i.e.*, a shock wave. In this sense the occurrence of such critical points indicates the necessary appearance of shock waves.

The phenomenon of critical points was found first by M. and F. Clauser who, on the suggestion of the present author, calculated a family of exact solutions of the Eq. (70) and discussed the corresponding flow patterns. They investigated especially the flow bounded by two inclined straight walls. Recently, F. Ringleb, by a method similar to that employed by the Clausers, found exact flow patterns of compressible fluids. He presents in his paper one example which is especially instructive as to the question of the first appearance of shock waves and will be briefly discussed here.

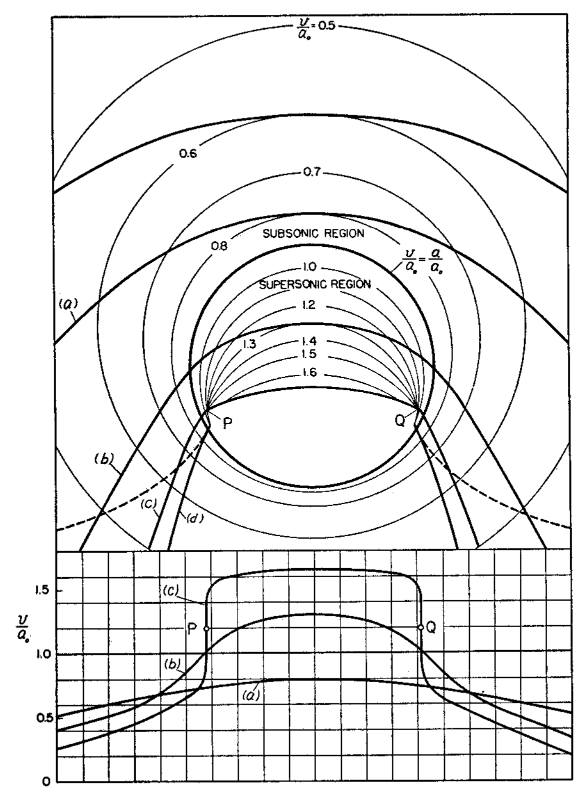

FIG. 20. Flow in a curved convergent-divergent nozzle (F. Ringleb).

Fig. 20 shows a kind of nozzle which is essentially a curved convergent-divergent nozzle. The boundaries are exact streamlines of a compressible fluid obtained in the manner explained above. The velocity is zero at the entrance and at the exit at infinity. The lines of constant velocity—which happen to be exact circles—show that the flow is partly subsonic, partly supersonic. Consider first the nozzle bounded by the lines (a) and (b). The velocity distribution along the two walls is given by the lines (a) and (b) in the lower part of Fig. 20. It is seen that the flow is perfectly smooth and continuous. However, if the nozzle is bounded by the lines (a) and (c) the velocity has infinite slopes at the points P and Q. If one plots another streamline proceeding forward inside, it is found that the streamline stops at a certain point and turns back, crossing the nozzle; this is, of course, physically impossible.

It is remarkable that the maximum velocity reached at the wall (c); *i.e.*, in the limiting case, is as high as $1.67a_0$ (where a_0 is the velocity of sound at rest) and that the velocity at the point where a discontinuity first appears in the flow is equal to $1.20a_0$.

The German author does not discuss further the conditions for the singular behavior of the streamlines. However, the equations developed in the section on the Hodograph Method of this paper make it possible to determine a condition for the occurrence of an infinite acceleration. Obviously the acceleration in the direction of the flow is equal to $(\partial v/\partial s)(ds/dt) = (\partial v/\partial s)v = (\partial v/\partial \varphi)v^2$. Hence, if v has a finite value, the acceleration can be infinite only if $\partial v/\partial \varphi = \infty$. Now according to Eq. (48)

$$\partial v/\partial \varphi = -(1/\Delta)(\partial \psi/\partial \theta)$$

where $\Delta = (\partial \varphi/\partial v)(\partial \psi/\partial \theta) - (\partial \varphi/\partial \theta)(\partial \psi/\partial v)$ is the so-called Jacobian previously discussed. If the case $\partial \psi/\partial \theta = \infty$ is excluded for the time being, $\partial v/\partial \varphi$ can be infinite only when

$$\Delta = (\partial \varphi/\partial v)(\partial \psi/\partial \theta) - (\partial \varphi/\partial \theta)(\partial \psi/\partial v) = 0 \quad (71)$$

Substituting the values of $\partial \varphi/\partial v$ and $\partial \varphi/\partial \theta$ from the Eq. (50) one obtains the condition

$$(1 - M^2)(\partial \psi/\partial \theta)^2 + v^2(\partial \psi/\partial v)^2 = 0 \quad (72)$$

The expression on the left side is obviously always positive if $M < 1$. Therefore, the condition (72) can be only satisfied if $M > 1$; *i.e.*, if the flow is supersonic. Eq. (72) has a simple geometrical meaning. Denote the angle of inclination between a circle $v = $ const. and the tangent to the line $\psi = $ const. at an arbitrary point of the hodograph plane by β. Then

$$\tan \beta = -\left(\frac{1}{v}\frac{\partial \psi}{\partial \theta}\right) \Big/ \frac{\partial \psi}{\partial v} \quad (73)$$

Hence, from Eq. (72) follows

$$\tan^2 \beta = 1/(1 - M^2) \quad (74)$$

In the theory of supersonic flow the angle $\sin^{-1}(1/M) = \alpha$ is called Mach's angle; it is equal to the angle between the velocity vector and the so-called stationary wave-front produced by a small perturbation applied to the fluid. Hence, according to Eq. (74), $\beta = \pm \alpha$. The lines intersecting the circles $v = $ const. at the angle α are called the *real characteristics* of the differential Eq. (70) and for given constant γ are epicycloids. Hence, infinite acceleration can occur at points where the curve $\psi = $ const. in the hodograph plane is tangen-

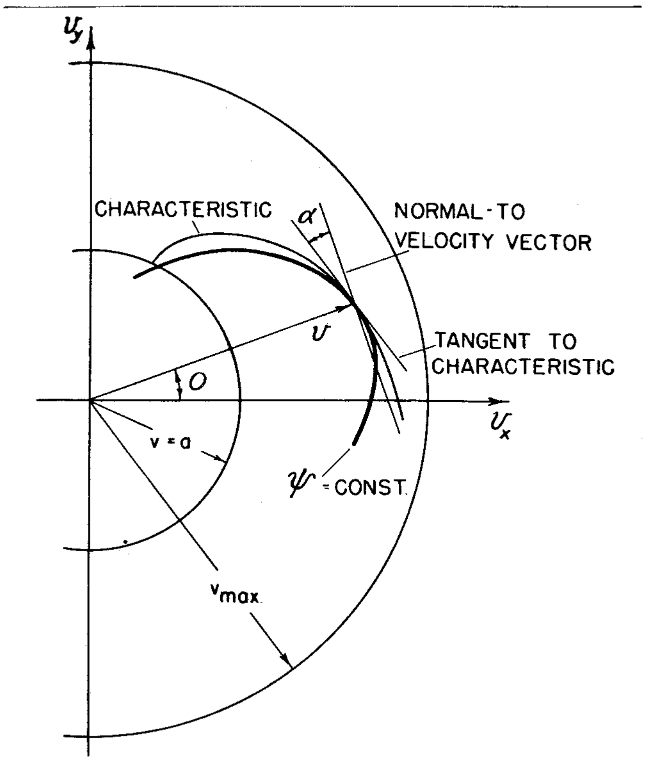

FIG. 21. Condition of infinite acceleration and breakdown of isentropic irrotational flow.

tial to a characteristic curve of the differential Eq. (70)* (Fig. 21).

The direction of the tangent to a ψ = const. curve indicates the direction of the infinitesimal change of the velocity vector; *i.e.*, the direction of the resultant acceleration, in the sense that the angle between the velocity and acceleration vectors is the same in the hodograph and physical planes. Hence, at the point where infinite acceleration occurs the angle between the resultant acceleration and the normal to the velocity is equal to the Mach angle. Applying this result to the case that the velocity along a streamline changes from subsonic to supersonic and back to subsonic again one can show that in order to avoid infinite acceleration the angle between the acceleration and the normal must be smaller than the local Mach angle in the entire supersonic range. Further investigations are necessary to expand the physical meaning and importance, if any, of this finding.

In Fig. 22 streamlines for the flow pattern shown in Fig. 20 are plotted in the hodograph plane. Fig. 23 refers to the velocity distribution found experimentally for the N.A.C.A. 4412 section at various Mach numbers. The lines are plots of the "zero streamline"

* The fact that the singular behavior of the streamlines occurs at points where streamline and characteristics in the hodograph plane have the same direction was first found by the Clausers. The condition (71) is mentioned in the paper of Ringleb. It appears that if a streamline is tangent to a characteristic in the hodograph plane, in the physical plane the characteristics (wavelets) of the same family meet; their envelop probably introduces the first discontinuous wave front.

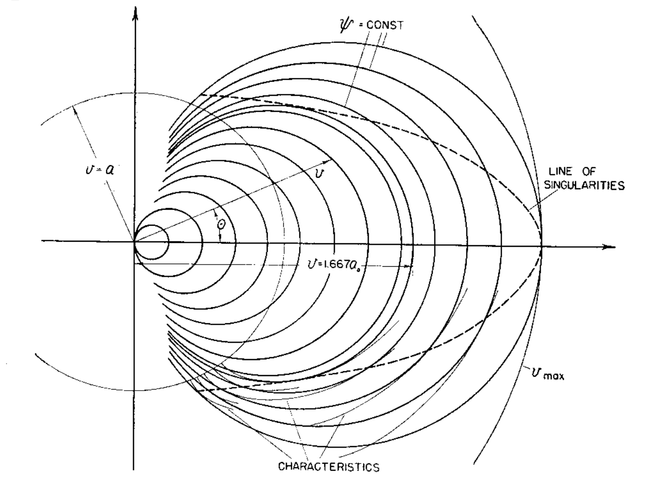

FIG. 22. Streamlines and characteristics in hodograph plane for a curved convergent-divergent nozzle.

(upper boundary) in the hodograph plane; in other words the velocity vectors occurring along the boundary are plotted as to magnitude and direction and their end points connected by a curve. It is seen that the line at a point close to the point where the shock wave occurs is tangential to characteristics passing through the same point. However, this coincidence might be only accidental because the velocity distribution is greatly changed by the shock wave, whereas the theorem proved above refers to the first occurrence of an infinite slope of the pressure, but not to a finite pressure jump.

Flow patterns with continuous transition of the velocity of sound in both directions have been calculated before by G. I. Taylor, W. Tollmien and recently by H. Görtler who investigated the flow along a wave-shaped solid wall by computing a second approximation on the lines of the Glauert-Prandtl method.

The author does not wish to exaggerate the importance of the above considerations. It is merely shown that in the case of an ideal compressible fluid continuous, shockless flow patterns can be constructed in which the velocity of flow reaches magnitudes as great as 1.6

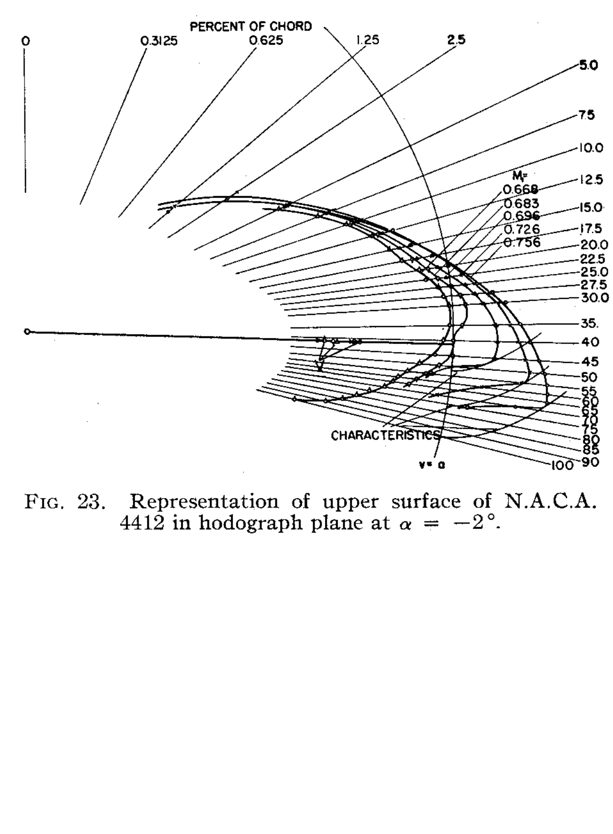

FIG. 23. Representation of upper surface of N.A.C.A. 4412 in hodograph plane at $\alpha = -2°$.

times the velocity of sound and becomes subsonic again. However, this does not prove yet that such flow patterns are really possible in a real fluid. Furthermore, it is possible that the flow with shock wave is more stable than the continuous flow pattern. If a body moves with supersonic velocity in a compressible fluid, shock waves appear unavoidable, *i.e.*, flying with supersonic velocity will always involve wave resistance in addition to the frictional drag. However, it appears that careful theoretical and experimental research might be able to push the velocity of flying nearer to the velocity of sound than is possible now. The mere fact that the air passes over a wing with supersonic velocity does not necessarily involve the occurrence of a compressibility burble and energy loss by shock wave.

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